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Explicit Equations to Transform from Cartesian to Elliptic Coordinates

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Abstract: Explicit equations are obtained to convert Cartesian coordinates to elliptic coordinates, based on which a function in elliptic coordinates can be readily mapped in physical space. Application to Kirchhoff vortex shows that its elliptical core induces two counter-rotating irrotational eddies.

Keywords: Elliptic Coordinates, Cartesian Coordinates, Kirchhoff Vortex

1. Introduction

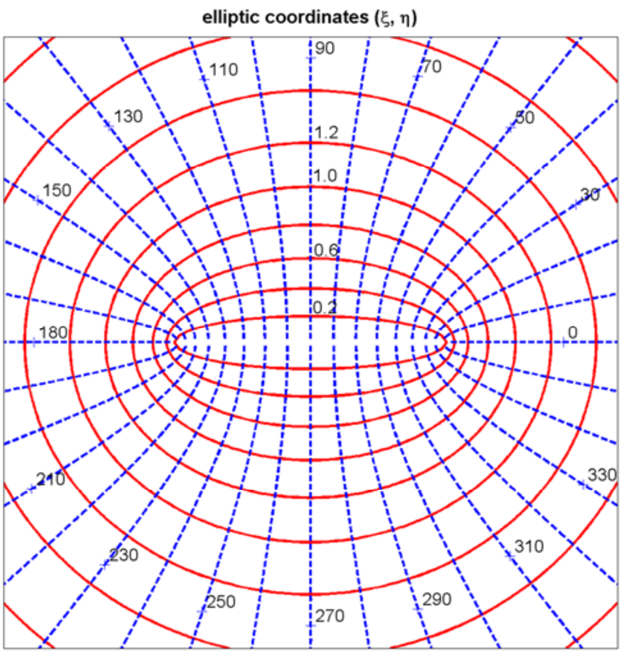

Figure 1. *Coordinate lines for an elliptic coordinate system with* $a=1, b=0.5$.

The elliptical coordinate system (ξ,η) as a two-dimensional orthogonal coordinate system has many dynamical and engineering applications, such as Kirchhoff vortex [1], insect aerodynamics [2], hydrodynamic wave diffraction [3], and theoretical physics [4]. Its coordinate lines are confocal ellipses and hyperbolae and the transformation from elliptic to Cartesian coordinates is given by

$$\begin{aligned} &x = c\cosh(\xi)\cos(\eta) \\ &y = c\sinh(\xi)\sin(\eta) \\ &c^2 = a^2 - b^2 \\ &\xi \ge 0, \quad 0 \le \eta < 2\pi \end{aligned} \tag{1}$$

where a and b denote the semi-major and semi-minor axes of the ellipse and c is the elliptical eccentricity (Figure 1).

In the meantime, explicit equations to transform from Cartesian to elliptic coordinates have not been found in the existing literature [5, 6, 7]. Such a conversion relation would be useful in mapping an elliptic-coordinate solution, for example the flow field of Kirchhoff vortex, in physical space.

2. Cartesian to Elliptic Coordinates

In order to invert the functional relation (1), we first eliminate ξ and have

$$\frac{x^2}{\cos^2(\eta)} - \frac{y^2}{\sin^2(\eta)} = c^2$$

which means curves of constant η are hyperbolae. The focus distance is c and the eccentricity is $e = \sec(\eta)$.

Let $p = \sin^2(\eta)$, we have

$$\frac{x^2}{1-p}-\frac{y^2}{p}=c^2$$

which becomes

$$c^2p^2+(x^2+y^2-c^2)p-y^2=0 \tag{2}$$

Then eliminating η from (1) we have

$$\frac{x^2}{\cosh^2(\xi)}+\frac{y^2}{\sinh^2(\xi)}=c^2$$

It shows that curves of constant ξ are ellipses. The focus distance is c and the eccentricity is $e=\cosh^{-1}(\xi)$.

Let $q=-\sinh^2(\xi)$, we have

$$\frac{x^2}{1-q}-\frac{y^2}{q}=c^2$$

which leads to

$$c^2q^2+(x^2+y^2-c^2)q-y^2=0 \tag{3}$$

It is essentially the same as (2). Therefore (p,q) constitute the two roots of a quadratic equation. Since $0\le p\le 1$, $q\le 0$, we have $p\ge q$, and the two roots are

$$p=\frac{-B+\sqrt{B^2+4c^2y^2}}{2c^2},\quad q=\frac{-B-\sqrt{B^2+4c^2y^2}}{2c^2} \tag{4}$$

in which $B=x^2+y^2-c^2$.

From the definition of p we obtain

$$\eta_0=\arcsin(\sqrt{p}) \tag{5}$$

It has four cases depending on which quadrant the Cartesian point (x,y) is located, i.e.,

$$\begin{aligned}
\eta&=\eta_0, && x\ge 0, y\ge 0\\
\eta&=\pi-\eta_0, && x<0, y\ge 0\\
\eta&=\pi+\eta_0, && x\le 0, y<0\\
\eta&=2\pi-\eta_0, && x>0, y<0
\end{aligned} \tag{6}$$

Based on the definition of q, we can solve ξ from quadratic equation

$$e^{4\xi}+(4q-2)e^{2\xi}+1=0,$$

which has two roots

$$e^{2\xi}=1-2q\pm 2\sqrt{q^2-q}$$

Since $q\le 0$, both roots are real and denoted as (ξ_1,ξ_2). They clearly satisfy $e^{2\xi_1}\cdot e^{2\xi_2}=1$, which leads to $\xi_2=-\xi_1<0$. Because in elliptical coordinates only non-negative ξ value is considered, we obtain

$$\xi=\frac{1}{2}\ln(1-2q+2\sqrt{q^2-q}) \tag{7}$$

Eqs. (4-7) are explicit equations to derive elliptic coordinates from Cartesian grid. They can easily be realized via computation software such as Matlab.

3. Application to Kirchhoff Vortex

Kirchhoff vortex is a rotating elliptical region of uniform vorticity ω embedded in an irrotational ideal fluid [8]. It is the simplest example of non-smooth weak solutions to the Euler equations and has wide application in vortex dynamics [9-11]. It has a discontinuity of vorticity across its elliptical boundary

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where a and b are semi-major and semi-minor axes, respectively. The boundary corresponds to contour $\xi=\operatorname{arcsinh}(b/c)$, and the vortex rotates with constant angular velocity

$$\Omega=\frac{ab}{(a+b)^2}\omega$$

around the origin. From Act. 159 of Lamb [1], the streamfunction outside the core is

$$\psi=\frac{1}{4}\Omega(a+b)^2e^{-2\xi}\cos(2\eta)+\frac{1}{2}\omega ab\xi \tag{8}$$

The instantaneous streamlines in a unsteady flow are given by the curves $\psi=const$. In a rotating frame with angular velocity Ω, Kirchhoff vortex looks steady and its streamfunction outside the elliptic core is related to the inertial-frame streamfunction (8) by

$$\psi_R=\psi-\frac{1}{2}\Omega[x(\xi,\eta)^2+y(\xi,\eta)^2] \tag{9}$$

From (9) it is straightforward to make a conformal mapping of steady streamfunction on a uniform mesh of elliptic coordinates (Figure 2).

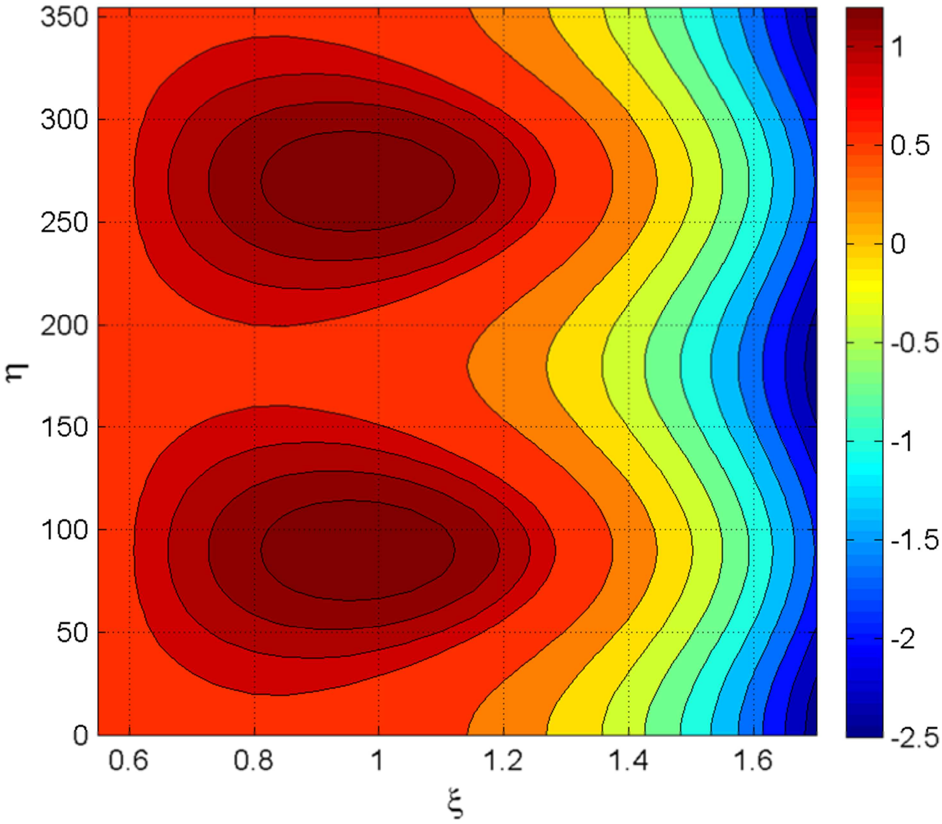

Figure 2. *Mapping of streamfunction (9) on a uniform mesh of* (ξ, η), *with* $a = 1, b = 0.5, \omega = 10$.

In order to view the flow field in physical space, we use the conversion equations (4-7) to plot streamfunction (9) on a uniform Cartesian mesh in Figure 3, which shows the elliptical core of Kirchhoff vortex induces two irrotational eddies that rotate in opposite directions.

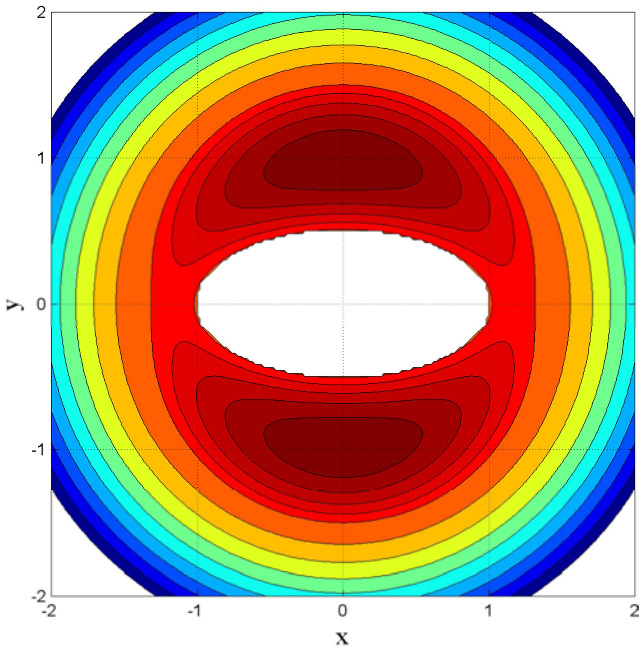

Figure 3. *Mapping of streamfunction (9) on a uniform mesh of xy-plane, with* $a = 1, b = 0.5, \omega = 10$. *The boundary of vortex core corresponds to* $\xi = 0.55$.

4. Conclusion

This study obtains explicit equations that convert Cartesian coordinates to elliptic coordinates. The conversion relation can be easily realized with computer software and used to map a known function of elliptic-coordinates, such as the streamfunction of Kirchhoff vortex, on a uniform Cartesian mesh.

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Matlab mfile scripts

```
function [zeta,eta]=xy2el(x,y,a,b)
%---------------------------------------------------------------------------
% Conversion from Cartesian mesh (x,y) to elliptic coordinates (zeta,eta)
% zeta (>=0), eta [0 2pi)
% a -- semi-major axis
% b -- semi-minor axis
% c -- elliptic eccentricity
% (x,y,zeta,eta) same-size matrix
% Author: Che Sun (CAS, 2016)
% Source: http://eprint.las.ac.cn/abs/201611.00721
%---------------------------------------------------------------------------
c2=(a^2-b^2);c=2*c2;
x2=x.^2; y2=y.^2;
B=x2+y2-c2;
del2=B.^2+2*c*y2; del=sqrt(del2);
q=(-B+del)/c; q=sqrt(q); et0=asin(q);
eta=x;

i=find(x>=0&y>=0); eta(i)=et0(i);
i=find(x<0&y>=0); eta(i)=pi-et0(i);
i=find(x<=0&y<0); eta(i)=pi+et0(i);
i=find(x>0&y<0); eta(i)=2*pi-et0(i);

p=(-B-del)/c;
del2=p.^2-p; del=sqrt(del2);
zeta=log(1-2*p+2*del)/2;% only keep the positive root


function [x,y]=el2xy(zeta,eta,a,b)
%---------------------------------------------------------------------------
% Conversion from elliptic coordiantes (zeta,eta) to Cartesian coordinates
% zeta (>=0), eta [0 2pi)
% a -- semi-major axis
% b -- semi-minor axis
% c -- elliptic eccentricity
% (x,y,zeta,eta) same-size matrix
% Author: Che Sun (CAS, 2016)
% Source: http://eprint.las.ac.cn/abs/201611.00721
%---------------------------------------------------------------------------
c2=a^2-b^2; c=sqrt(c2);
x=c*cosh(zeta).*cos(eta);
y=c*sinh(zeta).*sin(eta);
```